

The number of relatively r -prime k -tuple integers

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Abstract

For a fixed integer $r \geq 1$, we say k -tuple integers (x_1, \dots, x_k) are relatively r -prime if there exists no prime p such that all k integers is multiple of p^r . Benkoski proved that the number of relatively r -prime k -tuple integers in $[1, x]^k$ is $x^k/\zeta(rk) + (\text{Error term})$ [2]. We showed that the exact order of error term is x^{k-1} for $rk \geq 3$ and $k \neq 1$.

0 Introduction

From 1800's many results about the distribution of special lattice points were shown. F. Mertens proved that the density of the set of coprime pairs of integers is $1/\zeta(2)$ in 1874 [3]. And this result was extended to k -tuple integers by D. N. Lehmer [5]. On the other hand, Gegenbauer proved that the probability that an integer is r -free is $1/\zeta(r)$ in 1885 [4]. As a generalization of these result, S. J. Benkoski proved that the density of the set of relatively r -prime k -tuple integers is $1/\zeta(rk)$ in 1976 [2].

In [1], we computed the number of coprime k -tuple integers in $[-x, x]^k$ and the exact order of magnitude of its error term is x^{k-1} for all $k \geq 3$. We will generalize this result to relatively r -prime k -tuple integers. For fixed $r \geq 1$ let $(x_1, \dots, x_k)_r$ be integer n such that n is the greatest common factor of the form n^r for integer $n \geq 1$. When $r = 1$, $(x_1, \dots, x_k)_1$ means the great common divisor of x_1, \dots, x_k i.e. $\gcd(x_1, \dots, x_k)$. And let $V_k^r(x)$ denote the number of k -tuple integers (x_1, \dots, x_k) such that $(x_1, \dots, x_k)_r = 1$ and $|x_i| \leq x$ for all $i = 1, \dots, k$. When $r = 1$, $V_k^1(x)$ means the number of visible lattice points in $[-x, x]^k$ and $k = 1$ a half of $V_1^r(x)$ means the number of r -free positive integers $\leq x$. And we let $E_k^r(x)$ denote the error term, i.e. $E_k^r(x) = V_k^r(x) - (2x)^k/\zeta(rk)$.

In this paper, we compute $V_k^r(x)$ by following the ways of [1], so we get a generating function of $V_k^r(x)$ for a fixed positive integer r and the exact order of $E_k^r(x)$ is x^{k-1} for $rk \geq 3$ and $k \neq 1$.

1 Benkoski's result

To consider the exact order of $E_k^r(x)$, we use S. J. Benkoski result. He considered that the number of r -prime k -tuple integers (x_1, \dots, x_k) such that $1 \leq |x_i| \leq x$ for all $i = 1, \dots, k$ by using a general Jordan totient function $J_k^r(n)$ (Theorem 3, 4 and 5 in [2]). The general Jordan totient function is defined as follows.

Definition 1.1 (Definition of Chapter 2. [2]). *Let $r \geq 1$, $k \geq 1$ are integers.*

$$J_k^r(n) := |\{(x_1, \dots, x_k) \in \mathbb{Z}^k \mid (x_1, \dots, x_k, n)_r = 1, 1 \leq x_i \leq n \ (1 \leq i \leq k)\}|.$$

For $k = 0$ we define

$$J_0^r(n) := \begin{cases} 1 & (n \text{ is } r\text{-free}), \\ 0 & (\text{otherwise}). \end{cases}$$

If $r = 1$ then $J_k^1(n)$ is ordinary Jordan totient function and if $r = k = 1$ then $J_1^1(n)$ is the Euler totient function. We know that general Jordan totient function $J_k^r(n)$ has Dirichlet product and Euler product expansion,

$$J_k^r(n) = \sum_{d^r | n} \mu(d) \left(\frac{n}{d^r}\right)^k = n^k \prod_{\substack{p^r | n \\ p: \text{prime}}} \left(1 - \frac{1}{p^{rk}}\right).$$

When n is r -free, the product is empty and assigned to be the value 1. This formula is proved as well as an analogue statement of Euler totient function.

Benkoski considered only positive integers in his paper [2]. But considering the sign of component of (x_1, \dots, x_k) , we obtain the following asymptotic formula from Benkoski's result.

$$V_k^r(x) = \frac{2^k}{\zeta(rk)} x^k + \begin{cases} O(x \log x) & (r = 1 \text{ and } k = 2), \\ O(x^{1/r}) & (r \geq 2 \text{ and } k = 1), \\ O(x^{k-1}) & (\text{otherwise}). \end{cases}$$

2 The partial sums of the general Jordan totient function

In this paper, we use the Ω symbol introduced by G.H. Hardy and J.E. Littlewood. This symbol is defined as follows:

$$f(x) = \Omega(g(x)) \stackrel{\text{def}}{\iff} \limsup_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| > 0.$$

If there exists a function $g(x)$ such that $f(x) = O(g(x))$ and $f(x) = \Omega(g(x))$ then the exact order of $f(x)$ is $g(x)$. In [1], We calculated the exact order of $E_k^1(x)$ by using the following theorem.

Theorem 2.1 (Lemma 4.2. of [1]). *For $r = 1$ and $k \geq 3$*

$$\sum_{n \leq x} J_{k-1}^1(n) = \frac{x^k}{k\zeta(k)} + \Omega(x^{k-1})$$

We follow this way to consider the exact order of $E_k^r(x)$ for $r \geq 1$. So we consider an asymptotic formula for the partial sums of $J_{k-1}^r(n)$. Because the general Jordan totient function has Dirichlet product, by applying same argument of the Chapter 4 in [1], we get following equation:

$$\sum_{n \leq x} J_{k-1}^r(n) = \frac{1}{k} \sum_{j=0}^{k-1} \binom{k}{j} B_j \sum_{d^r \leq x} \mu(d) \left(\frac{x}{d^r} - \left\{ \frac{x}{d^r} \right\} \right)^{k-j},$$

where $\{x\}$ is the fractional part of x and B_0, B_1, B_2, \dots are Bernoulli numbers. (Note. We use the second Bernoulli number, i.e. $B_1 = \frac{1}{2}$.) To evaluate this sum, we will extend Lemma 4.1. of [1] to $r \geq 2$ case.

Lemma 2.2. *Let $\{x\}$ be the fractional part of x . If $rk \geq 2$,*

$$\sum_{d^r \leq x} \mu(d) \left(\frac{x}{d^r} \right)^k \left\{ \frac{x}{d^r} \right\} = \Omega(x^k).$$

Proof. It suffices to show that $\sum_{d^r \leq x} \frac{\mu(d)}{d^{rk}} \left\{ \frac{x}{d^r} \right\} \leq M < 0$ for infinity many values of x and some negative M . The case of $r = 1$ is proved in [1], so it suffices to consider $r \geq 2$.

If $rk \geq 4$, let x be a integer such that $x \equiv 2^r - 1 \pmod{2^r}$ and greater than or equal to 3^r ,

$$\begin{aligned} \sum_{d^r \leq x} \frac{\mu(d)}{d^{rk}} \left\{ \frac{x}{d^r} \right\} &= \sum_{d \leq \sqrt[r]{x}} \frac{\mu(d)}{d^{rk}} \left\{ \frac{x}{d^r} \right\} \\ &= -\frac{2^r - 1}{2^{r(k+1)}} + \sum_{3 \leq d \leq \sqrt[r]{x}} \frac{\mu(d)}{d^{rk}} \left\{ \frac{x}{d^r} \right\}. \end{aligned}$$

Since $\mu(d) = 1, 0, -1$ and $\left\{ \frac{x}{d^r} \right\} \leq 1$,

$$\begin{aligned} \sum_{d^r \leq x} \frac{\mu(d)}{d^{rk}} \left\{ \frac{x}{d^r} \right\} &< -\frac{1}{2^{rk}} + \frac{1}{2^{r(k+1)}} + \sum_{3 \leq d \leq \sqrt[r]{x}} \frac{1}{d^{rk}} \\ &< -\frac{1}{2^{rk}} + \frac{1}{2^{r(k+1)}} + \zeta(rk) - 1 - \frac{1}{2^{rk}}. \end{aligned}$$

When $rk \geq 4$ we know that $\zeta(rk) - 1 - \frac{1}{2^{rk}} < \frac{1}{2^{rk+1}}$, so we get

$$\sum_{d^r \leq x} \frac{\mu(d)}{d^{rk}} \left\{ \frac{x}{d^r} \right\} < -\frac{1}{2^{rk+1}} + \frac{1}{2^{r(k+1)}} < 0,$$

since $r \geq 2$.

So for $rk \geq 4$ the lemma follows.

Suppose that $(r, k) = (2, 1)$ or $(r, k) = (3, 1)$ and $x = m^2 \prod_{p \leq 100} p^r$, where the product is extended over all odd primes less than 100 and m isn't a multiple of 2 and p .

Then,

$$\sum_{d^r \leq x} \frac{\mu(d)}{d^r} \left\{ \frac{x}{d^r} \right\} = \sum_{d=1}^{100} \frac{\mu(d)}{d^r} \left\{ \frac{x}{d^r} \right\} + \sum_{d=101}^{x^{1/r}} \frac{\mu(d)}{d^r} \left\{ \frac{x}{d^r} \right\}.$$

Since $\mu(d) = 1, 0, -1$ and $\left\{ \frac{x}{d^r} \right\} < 1$,

$$\sum_{d^r \leq x} \frac{\mu(d)}{d^r} \left\{ \frac{x}{d^r} \right\} < -\frac{1}{2^{r+1}} + \sum_{d=3}^{100} \frac{\mu(d)}{d^r} \left\{ \frac{x}{d^r} \right\} + \sum_{d=101}^{\infty} \frac{1}{d^r}.$$

Now we estimate how fast second sum grows. When $r = 2$ we obtain

$$\begin{aligned} \sum_{d=3}^{100} \frac{\mu(d)}{d^2} \left\{ \frac{x}{d^2} \right\} &= \sum_{p=\text{prime}}^{47} \frac{1}{(2p)^2} \frac{1}{4} - \frac{1}{4} \left(\frac{1}{30^2} + \frac{1}{42^2} + \frac{1}{66^2} + \frac{1}{78^2} + \frac{1}{70^2} \right) \\ &< \frac{1}{50}. \end{aligned}$$

On the other hand, when $r = 3$,

$$\sum_{d=3}^{100} \frac{\mu(d)}{d^3} \left\{ \frac{x}{d^3} \right\} = \sum_{p=\text{prime}}^{47} \frac{1}{(2p)^3} \frac{\bar{p}}{8} - \frac{1}{8} \left(\frac{7}{30^3} + \frac{5}{42^3} + \frac{1}{66^3} + \frac{7}{78^3} + \frac{3}{70^3} \right),$$

where $\bar{p} \equiv p \pmod{8}$ and $0 \leq \bar{p} < 8$.

$$\sum_{d=3}^{100} \frac{\mu(d)}{d^3} \left\{ \frac{x}{d^3} \right\} < \frac{2}{5} \times \frac{1}{10^2}.$$

From this result and we have $\sum_{d=101}^{\infty} \frac{1}{d^r} \leq \frac{1}{100^{r-1}}$, we find

$$\begin{aligned} \sum_{d^r \leq x} \frac{\mu(d)}{d^r} \left\{ \frac{x}{d^r} \right\} &< -\frac{1}{2^{r+1}} + \frac{2}{5} \times \frac{1}{10^{r-1}} + \frac{1}{100^{r-1}} \\ &< -\frac{1}{20}, \end{aligned}$$

so for $(r, k) = (2, 1)$ or $(r, k) = (3, 1)$ the lemma follows.

This completes the proof of the lemma. □

As we remarked, the partial sums of $J_{k-1}^r(n)$ is equal to

$$\frac{1}{k} \sum_{j=0}^{k-1} \binom{k}{j} B_j \sum_{d^r \leq x} \mu(d) \left(\frac{x}{d^r} - \left\{ \frac{x}{d^r} \right\} \right)^{k-j}.$$

We computed the order of the sum of $\mu(d) \left(\frac{x}{d^r} \right)^{k-1} \left\{ \frac{x}{d^r} \right\}$ for all $rk \geq 2$ in Lemma 2.2. Next we will consider the order of principal term $\sum_{d^r \leq x} \mu(d) \frac{x^k}{d^{rk}}$ of the partial sums of $J_k^r(n)$ in the Proposition 2.3.

Proposition 2.3. *For $rk \geq 2$, $\sum_{d^r \leq x} \mu(d) \frac{x^k}{d^{rk}} = \frac{x^k}{\zeta(rk)} + O(x^{1/r})$.*

Proof. We have

$$\sum_{d^r \leq x} \mu(d) \frac{x^k}{d^{rk}} = \sum_{d \leq x^{1/r}} \mu(d) \frac{x^k}{d^{rk}}.$$

We know that $\sum_{d \leq x} \frac{\mu(d)}{d^s} = \frac{1}{\zeta(s)} + O(x^{-s+1})$ for $s > 1$. (For the details for the proof of this result, one can see Theorem 11.7 of Apostol's book [6]).

Use this asymptotic formula,

$$\begin{aligned} \sum_{d^r \leq x} \mu(d) \frac{x^k}{d^{rk}} &= x^k \left(\frac{1}{\zeta(rk)} + O\left((x^{1/r})^{-rk+1}\right) \right), \\ &= \frac{x^k}{\zeta(rk)} + O(x^{1/r}). \end{aligned}$$

This proposition holds. □

We note that for all i

$$\left| \sum_{d^r \leq x} \frac{\mu(d)}{d^{rj}} \left\{ \frac{x}{d^r} \right\}^i \right| \leq \sum_{d^r \leq x} \frac{1}{d^{rj}} = \begin{cases} \zeta(rj) + O(x^{1/r-j}) & (rj \geq 2), \\ \log x + \gamma + o(1) & (rj = 1), \end{cases}$$

where γ is Euler's constant, defined by the equation

$$\gamma = \lim_{n \rightarrow \infty} \left(\sum_{k=1}^n \frac{1}{k} - \log n \right).$$

As we considered above, we get an order of all terms in the partial sums of $J_k^r(n)$. Using this result, we get the following theorem.

Theorem 2.4. *For $rk \geq 3$ and $k \neq 1$,*

$$\sum_{n \leq x} J_{k-1}^r(n) = \frac{x^k}{k\zeta(rk)} + \Omega(x^{k-1}).$$

Proof. As already remarked, we know that

$$\sum_{n \leq x} J_{k-1}^r(n) = \frac{1}{k} \sum_{j=0}^{k-1} \binom{k}{j} B_j \sum_{d^r \leq x} \mu(d) \left(\frac{x}{d^r} - \left\{ \frac{x}{d^r} \right\} \right)^{k-j}.$$

Using the binomial theorem and the order of summation,

$$\sum_{n \leq x} J_{k-1}^r(n) = \frac{1}{k} \sum_{j=0}^{k-1} \binom{k}{j} B_j \sum_{i=0}^{k-j} (-1)^i \sum_{d^r \leq x} \mu(d) \left(\frac{x}{d^r} \right)^{k-j-i} \left\{ \frac{x}{d^r} \right\}^i$$

Combining the remark before of this Theorem with Lemma 2.2 and Proposition 2.3, we get

$$\begin{aligned} \sum_{n \leq x} J_{k-1}^r(n) &= x^k \sum_{d^r \leq x} \frac{\mu(d)}{d^{rk}} + \Omega(x^{k-1}) \\ &= \frac{x^k}{k\zeta(rk)} + \Omega(x^{k-1}). \end{aligned}$$

This proved the lemma.

□

3 Generating function of $V_k^r(x)$

In this section, we consider a generating function of $V_k^r(x)$. The case of $r = 1$ was considered in [1]. We will prove a generalisation of the case of $r = 1$ by following the method of Theorem 3.1. of [1].

Theorem 3.1. *Generating function of $V_k^r(x)$ is the following.*

$$\sum_{k=0}^{\infty} \frac{u^k}{k!} V_k^r(x) = \frac{1}{2u} (e^{(2X+1)u} - e^{(2X-1)u})$$

and

$$\sum_{k=0}^{\infty} u^{k+1} V_k^r(x) = \frac{1}{2} \log \frac{1 - (2X-1)u}{1 - (2X+1)u},$$

where $k \sum_{n \leq x} J_{k-1}^r(n)$ is replaced by X^k when $k \geq 1$, and X^0 are assigned to be the value 0.

Proof. We can show these results as well as Theorem 3.1. of [1]. It suffices to show that

$$V_k^r(x) = \frac{1}{2(k+1)} \{(2X+1)^{k+1} - (2X-1)^{k+1}\}.$$

After change of functions $J_k(n)$ into $J_k^r(n)$ in proof of Theorem 3.1. of [1], we can compute $V_k^r(x)$ in same combinatorial way.

Hence we obtain following equation

$$V_k^r(x) = \sum_{i=0}^{k-1} \binom{k}{i} 2^{k-i} \left(\sum_{n \leq x} \sum_{j=0}^{k-i-1} (-1)^{k-i-1-j} \binom{k-i}{j} J_j(n) \right).$$

Applying the binomial theorem and changing the order of summation of it, we show

$$V_k^r(x) = \frac{1}{2(k+1)} \{(2X+1)^{k+1} - (2X-1)^{k+1}\}.$$

This proves the theorem.

□

4 The exact order of magnitude of $E_k^r(x)$

We showed that $V_k^r(x)$ is finite linear combination of $\sum_{n \leq x} J_{k-1}^r(n)$ in last section.

Combining this result with Theorem 2.4, we get the exact order of magnitude of $E_k^r(x)$ as follows.

Theorem 4.1. *If $rk \geq 3$ and $k \neq 1$,*

$$E_k^r(x) = \Omega(x^{k-1}).$$

Proof. We can prove this theorem easily from Theorem 3.1.

From Theorem 3.1

$$\begin{aligned} V_k^r(x) &= \frac{1}{2(k+1)} \{(2X+1)^{k+1} - (2X-1)^{k+1}\} \\ &= (2X)^k + O(X^{k-2}). \end{aligned}$$

Applying Theorem 2.4, we find

$$V_k^r(x) = \frac{2^k}{\zeta(rk)} x^k + \Omega(x^{k-1}).$$

Hence $E_k^r(x) = \Omega(x^{k-1})$ for $rk \geq 3$ and $k \neq 1$

□

Combine Benkoski's result with this theorem, we prove that the exact order of magnitude of $E_k^r(x)$ is x^{k-1} , for all $rk \geq 3$ and $k \neq 1$.

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